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Locally C_6 graphs are clique divergent

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Abstract

The *clique graph* kG of a graph G is the intersection graph of the family of all maximal complete subgraphs of G. The iterated clique graphs k^nG are defined by $k^0G=G$ and $k^{n+1}G = kk^nG$. A graph G is said to be *k-divergent* if $|V(k^nG)|$ tends to infinity with n. A graph is *locally* C_6 if the neighbours of any given vertex induce an hexagon. We prove that all locally C_6 graphs are k-divergent and that the diameters of the iterated clique graphs also tend to infinity with n while the sizes of the cliques remain bounded. © 2000 Published by Elsevier Science B.V. All rights reserved.

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1. Introduction

If G is a (finite, unoriented, simple, connected and non-empty) graph, its *clique graph* kG is the intersection graph of the family of all *cliques* (maximal complete subgraphs) of G. Thus, two different cliques of G are adjacent in kG iff they share some common vertex. The *iterated clique graphs*, which constitute our subject, are defined by $k^0G=G$ and $k^{n+1}G=kk^nG$. We refer to [16,9] for the literature on iterated clique graphs. The graph G is *clique divergent* (or *k-divergent*) if the orders $o(k^nG)$ tend to infinity with n. The first examples of k-divergent graphs were obtained by Neumann-Lara; in particular, for $d \geqslant 3$, denote by $\mathcal{O}_d$ the d-dimensional octahedron, which considered as a graph can be described as the complement of a 1-factor of the complete graph K_{2d}. It has been proved that $k\mathcal{O}_d \cong \mathcal{O}_{2^{d-1}}$ and hence $\mathcal{O}_d$ is k-divergent [5,11].

If G and H are graphs, G is said to be *locally* H if the set of neighbours of every given vertex of G induces a subgraph isomorphic to H. These are

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graphs with a *constant link* in the terminology of [2]. In particular, we are interested in locally C_6 graphs.

The only (connected!) locally C_3 graph is the tetrahedron K_4, and its clique graph is the trivial graph, so K_4 is a special case of a *k-null* graph: some iterated clique graph is trivial.

The only locally C_4 graph is the octahedron $\mathcal{O}_3$, and we have just mentioned that it is k-divergent.

The only locally C_5 graph is the icosahedron, and since the 1970s the problem of deciding whether it is k-divergent has remained open [12,13].

There is an infinite number of locally C_6 graphs; they constitute a large part of the set of underlying graphs G of the maps (see [4]) of type $\{3,6\}$ in either the torus or the Klein bottle but they are not all: There is one such map in the torus with $G=K_7$ and, on the other hand, the minimum order for a locally C_6 graph is 12. In [9] we studied a family $\{G(r,s)\,|\,r,s\geqslant 4\}$ of locally C_6 graphs and we proved that they are k-divergent. The graphs $G(r,s)$ were the first known examples of k-divergent graphs G such that not only the orders $o(k^nG)$ but also the diameters $d(k^nG)$ tend to infinity with n, thus solving in the affirmative B. Hedman's problem 5 in [7]. The graphs $G(r,s)$ even had a stronger property: the *clique numbers* $\omega(k^nG)$ (the order of a biggest clique of k^nG) remain bounded as n increases. This property solved in the negative Neumann-Lara's problem 2 in [13], which asked if the k-divergence of G implies that the sequence of the clique numbers $\omega(k^nG)$ diverges. In this paper we will extend this results to all locally C_6 graphs and prove:

Theorem 1.1. *If G is a locally C_6 graph, then G is k-divergent and $d(k^nG)$ tends to infinity with n. Furthermore, $\omega(k^nG)$ remains bounded as n increases.*

That the boundedness of the clique numbers implies the divergence of the sequence of the diameters was proved in [9], but it will not be needed here.

Locally C_6 graphs have been already studied in other contexts. As one can see in [3], the orientable case appeared in connection with some of the groups of genus one, whose complete description was achieved by Dyck in the early eighties of the previous century. It seems to us in view of the pictures in [3] that our Proposition 4.1, which is taken for granted in [4], would already be clear and known more than a hundred years ago, but we have included a proof for the reader's convenience.

Let us conclude this introduction with some indications to the technique that will be used. The *triangular complex* of a graph G is the simplicial complex $\mathbb{K}=\mathbb{K}(G)$ whose vertices are those of G and whose simplexes are the vertices, edges and triangles of G. This is just the 2-skeleton of the *total* complex $G^{\uparrow}$ which has been studied by E. Prisner in the context of iterated clique graphs (see [14]). If G is locally C_t, each edge is contained in precisely two triangles, and the triangles containing any given vertex can be cyclically ordered in such a way that each one of them shares precisely one edge with that one following it in that cyclical order. This means that $\mathbb{K}$ is a triangulation of some closed surface, or, in other words, that the geometric realization

$|\mathbb{K}|$ of $\mathbb{K}$ is a closed topological surface. From now on, we will denote by the same symbol $\mathbb{K}$ both the combinatorial simplicial complex and its geometric realization, or rather the topological space with its fixed simplicial decomposition. If G is locally C_t with $t \in \{3,4,5\}$, then $\mathbb{K}$ is the sphere. If G is locally C_6, then $\mathbb{K}$ is either the torus or the Klein bottle: Indeed, if G has v vertices, e edges and t triangles, then $6v=2e=3t$, so $e=3v$ and $t=2v$; the Euler characteristic is then $\chi(\mathbb{K})=v-e+t=v-3v+2v=0$, and the classification theorem for closed surfaces implies that $\mathbb{K}$ is either a torus or a Klein bottle, according to whether it is orientable or not.

In Section 2 we will show that the proof of Theorem 1.1 can be reduced to the orientable case by considering a double covering of the Klein bottle by a torus. In Section 3 we will describe a graph-theoretical analogue of the topological regular covering maps. In Section 4 we will study the orientable case using the universal covering map from the plane to the torus. In order to do this last we will need to consider also infinite graphs.

We refer to [1] or [6] for graph theory and to [10,20] or [19] for topology.

2. Triangular covering maps

If G and H are graphs, a *homomorphism* $f: G \to G'$ is any adjacency-preserving vertex-map $f: V(G) \to V(G')$, i.e., $[u,v] \in E(G) \Rightarrow [f(u), f(v)] \in E(G')$. Since our graphs do not have loops, any homomorphism is injective when restricted to a complete subgraph of the domain, i.e., a subgraph with diameter at most one. If $v \in V(G)$, we denote by $N[v]$ the subgraph of G induced by v and its neighbours. Note that if $f: G \to G'$ is a homomorphism of graphs, then all the restrictions $f_| : N[v] \to N[f(v)]$ are also homomorphisms.

A homomorphism of graphs $p: \tilde{G} \to G$ is a *covering map* of graphs iff p satisfies the *unique path lifting property*: For any path $\gamma = (v_0, v_1, \ldots, v_l)$ in G and any vertex $\widetilde{v_0} \in p^{-1}(v_0)$ there exists a unique path $\tilde{\gamma} = (\widetilde{v_0}, \widetilde{v_1}, \ldots, \widetilde{v_l})$ in $\tilde{G}$ such that, with the obvious meaning, $p(\tilde{\gamma}) = \gamma$. Of course, this is equivalent to the *unique edge lifting property*. All the fibers $p^{-1}(v)$ of a covering map have the same cardinality: this is the *number of sheets* of p and when it is finite p is said to be *finite*. A covering map of graphs is injective when restricted to any subgraph with diameter at most two. Since our graphs are connected and non-empty, any covering map of graphs is surjective in vertices and in edges. In particular, note that all the restrictions $p_| : N[\tilde{v}] \to N[p(\tilde{v})]$ are embeddings of spanning subgraphs (vertex-bijective homomorphisms), and that this is an equivalent definition of a covering map.

Lemma 2.1. *Let* $p: \tilde{G} \to G$ *be a finite covering map with* h *sheets. Then*

$$d(G) \leqslant d(\tilde{G}) \leqslant h \cdot (2\,\mathrm{rad}(G) + 1) - 1.$$

Proof. The first inequality clearly holds for any vertex-surjective homomorphism. To see the second one, choose a central vertex $v \in V(G)$ and a spanning tree $T \leqslant G$ such that $v \in V(T)$ and $\mathrm{rad}(T) = \mathrm{rad}(G)$; thus, $d(T) \leqslant 2\,\mathrm{rad}(G)$. The unique path lifting property of p implies (is even equivalent to) the *unique tree lifting property*. Let $p^{-1}(v) = \{\widetilde{v_1}, \widetilde{v_2}, \ldots, \widetilde{v_h}\}$ and for $1 \leqslant i \leqslant h$ denote by $\widetilde{T_i}$ the lifting of T with root at $\widetilde{v_i}$. The $\widetilde{T_i}$ are all disjoint, isomorphic to T, and their union is a spanning forest of $\tilde{G}$ which extends to some spanning tree $\tilde{T} \leqslant \tilde{G}$. Thus we obtain that $d(\tilde{G}) \leqslant d(\tilde{T}) \leqslant h \cdot 2\,\mathrm{rad}(G) + h - 1$. $\square$

Remark 2.1. Using a similar technique, one can also prove in the situation of the preceding lemma that

$$\mathrm{rad}(G) \leqslant \mathrm{rad}(\tilde{G}) \leqslant \mathrm{rad}(G) + \left\lceil \frac{h-1}{2} \right\rceil \cdot (2\,\mathrm{rad}(G) + 1).$$

A covering map of graphs $p : \tilde{G} \to G$ is a *triangular covering map* if it also satisfies the (necessarily *unique*) *triangle lifting property*: for each triangle $[u, v, w]$ in G and each lifting $\tilde{u}$ of u, there exists a triangle $[\tilde{u}, \tilde{v}, \tilde{w}]$ lifting $[u, v, w]$ and containing $\tilde{u}$. A covering map of graphs is triangular if and only if the images of any two vertices lying apart at distance 3 are different. A homomorphism of graphs $p : \tilde{G} \to G$ is a triangular covering map if and only if all the restrictions $p_| : N[\tilde{v}] \to N[p(\tilde{v})]$ are isomorphisms of graphs: triangular covering maps and *local isomorphisms* of graphs are the same.

Triangular covering maps are well suited to our problem: If H is a graph and $p : \tilde{G} \to G$ is any such map, then G is locally H if and only if $\tilde{G}$ is locally H. On the other hand, the clique operator k induces a functor when restricted to the category of graphs and triangular covering maps, although of this we will only use the following proposition:

Proposition 2.2. *If* $p : \tilde{G} \to G$ *is a triangular covering map, define the vertex-map* $p_k : V(k\tilde{G}) \to V(kG)$ *by* $p_k(\tilde{Q}) = p(\tilde{Q})$. *Then* $p_k : k\tilde{G} \to kG$ *is also a triangular covering map and it has the same number of sheets as* p.

Proof. Note that any clique Q of any graph is contained in $N[v]$ for each $v \in Q$, and a subset Q of some $N[v]$ is a clique of the graph if and only if Q is a clique of $N[v]$ (we identify induced subgraphs with their vertex sets for notational simplicity). Thus, if $\tilde{Q}$ is a clique of $\tilde{G}$ and $\tilde{v} \in \tilde{Q}$, then $\tilde{Q}$ is a clique of $N[\tilde{v}]$, so $p(\tilde{Q})$ is a clique of $N[p(\tilde{v})]$ and hence of G: p sends cliques to cliques and so p_k is well defined.

If $[\tilde{Q}, \tilde{R}] \in E(\tilde{G})$, take $\tilde{v} \in \tilde{Q} \cap \tilde{R}$; then $\tilde{Q}$ and $\tilde{R}$ are different subsets of $N[\tilde{v}]$, so $p(\tilde{Q}) \neq p(\tilde{R})$ and $[p(\tilde{Q}), p(\tilde{R})] \in E(G)$. Thus p_k is a homomorphism.

Let $[Q, Q'] \in E(kG)$ and let $\tilde{Q} \in V(k\tilde{G})$ be any lifting of Q. Take a $v \in Q \cap Q'$ and lift it to a $\tilde{v} \in \tilde{Q}$. By hypothesis Q' can be uniquely lifted to a clique $\widetilde{Q'}$ which contains $\tilde{v}$, and so $[\tilde{Q}, \widetilde{Q'}] \in E(k\tilde{G})$. If $\widetilde{Q''}$ is any lifting of Q' to a neighbour of $\tilde{Q}$ in $k\tilde{G}$, then $\tilde{Q}''$ must contain both a lifting $\widetilde{v'}$ of v and a vertex $\tilde{u}$ of $\tilde{Q}$. The vertices $\tilde{v}, \tilde{u}, \widetilde{v'}$ induce a subgraph of diameter at most two, so $p(\tilde{v}') = p(\tilde{v})$ implies $\tilde{v} = \widetilde{v'}$ and then $\tilde{Q}' = \tilde{Q}$. This shows that p_k is a covering map.

Let $\tilde{Q}, \tilde{T} \in V(k\tilde{G})$ be such that $d(\tilde{Q}, \tilde{T}) = 3$ and let $(\tilde{Q}, \tilde{R}, \tilde{S}, \tilde{T})$ be a path. Let $\tilde{u} \in \tilde{Q} \cap \tilde{R}$, $\tilde{v} \in \tilde{R} \cap \tilde{S}$, and $\tilde{w} \in \tilde{S} \cap \tilde{T}$. Were it the case that $p(\tilde{Q}) = p(\tilde{T})$, there would exist a vertex $\tilde{u}' \in \tilde{T}$ such that $p(\tilde{u}') = p(\tilde{u})$. Since the path $(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{u}')$ lies in a subgraph of diameter at most three, we get that $\tilde{u}' = \tilde{u}$ and so $\tilde{Q} \cap \tilde{T} \neq \emptyset$, contradicting that $d(\tilde{Q}, \tilde{T}) = 3$. Thus we see that p_k is triangular.

Being local isomorphisms, triangular covering maps also satisfy the *unique clique lifting property*: If Q is a clique of G and $\tilde{v} \in p^{-1}(Q)$, then there exists a unique clique $\tilde{Q}$ of $\tilde{G}$ which contains $\tilde{v}$ and satisfies $p(\tilde{Q}) = Q$. From this, the equality of the number of sheets of p and p_k follows. $\square$

Corollary 2.3. *If $p : \tilde{G} \to G$ is a finite triangular covering map, then*

1. *$\tilde{G}$ is k-divergent if and only if G is k-divergent,*
2. *$\lim_{n\to\infty} d(k^n\tilde{G}) = \infty$ if and only if $\lim_{n\to\infty} d(k^n G) = \infty$,*
3. *$\omega(k^n\tilde{G}) = \omega(k^n G)$ for all n (even if p is not finite).*

Proof. (1) Follows from $\mathrm{o}(k^n\tilde{G}) = h \cdot \mathrm{o}(k^n G)$, where h is the number of sheets of p.

(2) uses Lemma 2.1 and $\mathrm{rad}(H) \leqslant d(H) \leqslant 2\,\mathrm{rad}(H)$ for any H.

(3) holds because we have a local isomorphism $p_{k^n} : k^n\tilde{G} \to k^n G$ for each n. $\square$

Let G be a finite locally C_6 graph such that the triangular complex $\mathbb{K} = \mathbb{K}(G)$ is a Klein bottle. Let $\pi : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{K}$ be a two-sheeted covering map from the two-dimensional torus to the Klein bottle. Using π to lift the triangulation of $\mathbb{K}$ to one of $\mathbb{S}^1 \times \mathbb{S}^1$, we obtain a triangulation $\tilde{\mathbb{K}}$ of the torus, and just as our graph G is the combinatorial 1-skeleton of $\mathbb{K}$ and $\mathbb{K} = \mathbb{K}(G)$, so the 1-skeleton of $\tilde{\mathbb{K}}$ is a graph $\tilde{G}$ such that $\mathbb{K}(\tilde{G}) = \tilde{\mathbb{K}}$.

The restriction to $V(\tilde{G})$ of the covering projection π yields a homomorphism of graphs $p : \tilde{G} \to G$, and the topological unique path lifting property of π implies the combinatorial unique path lifting property of p, which is then a covering map of graphs. That p satisfies the triangle lifting property is a consequence of our definition of $\tilde{\mathbb{K}}$ by lifting, because the lifting of any contractible curve in $\mathbb{K}$, for instance the *edge-path* $\gamma = (u, v, w, u)$ around any triangle of G, must be contractible in $\tilde{\mathbb{K}}$, and hence for any $\tilde{u} \in p^{-1}(u)$ the unique lifting $\tilde{\gamma}$ of γ starting at $\tilde{u}$ must be a closed curve, so the combinatorial lifting $\tilde{\gamma} = (\tilde{u}, \tilde{v}, \tilde{w}, \tilde{u})$ of γ under $p : \tilde{G} \to G$ is indeed a closed path, and hence $\tilde{u}, \tilde{v}, \tilde{w}$ induce a triangle in $\tilde{G}$.

Thus we have a triangular covering map with two sheets $p : \tilde{G} \to G$ and $\tilde{G}$ is also a locally C_6 graph. Since we have also that the triangular complex $\mathbb{K}(\tilde{G}) = \tilde{\mathbb{K}}$ is the torus, we see that the proof of Theorem 1.1 is reduced by Corollary 2.3 to the orientable case.

We have started with a topological covering map and then constructed a triangular covering map; let us also remark that, in a converse way, any triangular covering map $p : \tilde{G} \to G$ of graphs gives rise to a topological covering map $\mathbb{K}(p) : \mathbb{K}(\tilde{G}) \to \mathbb{K}(G)$. At the combinatorial level, $\mathbb{K}(p)$ is a simplicial map which is a covering map of complexes, and if the definition of these [17] looks more complicated than that of triangular covering maps, it is because we are forcing that in our complexes each triangle of the 1-skeleton bounds a 2-simplex.

3. Galois triangular covering maps

As suggested in the introduction, in order to solve the orientable case we will use a triangular covering map $p:\tilde{G} \to G$ with an infinite graph $\tilde{G}$. In this situation, statements (1) and (2) of Corollary 2.3 cease to be meaningful, and we will need to exploit further the structure of the maps used in this work: they are Galois triangular covering maps.

Given a homomorphism of graphs $p:\tilde{G} \to G$, the conditions 'p is a triangular covering map' and '$\mathbb{K}(p):\mathbb{K}(\tilde{G}) \to \mathbb{K}(G)$ is a topological covering map' are equivalent. Galois triangular covering maps will be those homomorphisms of graphs which correspond to regular (i.e., group-given) topological covering maps. In topology, it is required that a group of homeomorphisms of a space acts in a properly discontinuous way in order to ensure that the projection to the space of orbits is a covering map; the corresponding condition for a group of automorphisms of a graph will be called *admissibility*.

If $\Gamma \leqslant \mathrm{Aut}(G)$ is a group of automorphisms of G, denote by $\Gamma v = \{\gamma v \mid \gamma \in \Gamma\}$ the *Γ-orbit* of the vertex v. The *quotient graph* G/Γ has as vertices all the Γ-orbits of vertices of G, and two different Γ-orbits Γv and Γw are joined by an edge in G/Γ iff they contain adjacent representatives: there exist $\gamma v \in \Gamma v$ and $\gamma' w \in \Gamma w$ such that $[\gamma v, \gamma' w] \in E(G)$.

An automorphism $\gamma \in \mathrm{Aut}(G)$ is *coaffine* if $d(v, \gamma v) > 1$ for all $v \in V(G)$. If $\gamma \in \mathrm{Aut}(G)$ is coaffine, then $\gamma_k : kG \to kG$ is also a coaffine automorphism of kG (see [11]). A subgroup $\Gamma \leqslant \mathrm{Aut}(G)$ will be said to be *coaffine* if, apart from the identity, every element of Γ is coaffine. If this is the case, then the subgroup $\Gamma_k = \{\gamma_k \mid \gamma \in \Gamma\} \leqslant \mathrm{Aut}(kG)$ is also coaffine and $\Gamma_k \cong \Gamma$.

A subgroup $\Gamma \leqslant \mathrm{Aut}(G)$ will be called *admissible* if apart from the identity every element γ of Γ is *admissible* in the sense that $d(v, \gamma v) > 3$ for all vertices $v \in V(G)$. In particular, Γ admissible implies Γ coaffine. Given any Γ, we denote by $p_\Gamma : V(G) \to V(G/\Gamma)$ the natural projection defined by $p_\Gamma(v) = \Gamma v$; this is not always a homomorphism of graphs.

Lemma 3.1. *Let $\Gamma \leqslant \mathrm{Aut}(G)$ be coaffine. Then $p_\Gamma : G \to G/\Gamma$ is a homomorphism of graphs. Furthermore, p_Γ is a triangular covering map if and only if Γ is admissible.*

Proof. If $[v, w] \in E(G)$, then $\Gamma v \neq \Gamma w$ by coaffinity and hence $[\Gamma v, \Gamma w] \in E(G/\Gamma)$ by definition, so p_Γ is a homomorphism of graphs.

If p_Γ is a triangular covering map, it is injective when restricted to any subgraph of diameter at most 3, so $d(v, \gamma(v)) > 3$ for any $v \in V(G)$ and any $\gamma \in \Gamma$ different from the identity; thus, Γ is admissible.

If Γ is admissible, then p_Γ is vertex-injective when restricted to any subgraph of diameter at most 3. If $\Gamma v \in V(G/\Gamma)$ and $[\Gamma v, \Gamma w] \in E(G/\Gamma)$, by definition $[\gamma' v, \gamma'' w] \in E(G)$ for some $\gamma', \gamma'' \in \Gamma$. Hence, if we put $\gamma = (\gamma')^{-1}\gamma''$, the edge $[v, \gamma w]$ is a (necessarily unique) lifting of $[\Gamma v, \Gamma w]$ which starts at v. This proves unique path lifting and it follows that p_Γ is a triangular covering map. $\square$

Let $p : \tilde{G} \to G$ be a triangular covering map. We say that p is a *Galois triangular cover with group* Γ iff G can be identified with $\tilde{G}/\Gamma$ for some admissible $\Gamma \leqslant \mathrm{Aut}(\tilde{G})$ in such a way that p becomes the natural projection $p_\Gamma : \tilde{G} \to \tilde{G}/\Gamma$; this means that there exists an isomorphism $\varphi : \tilde{G}/\Gamma \to G$ such that $p = \varphi \circ p_\Gamma$, but one can think that $G = \tilde{G}/\Gamma$ and $p = p_\Gamma$. Galois triangular covers can be described as follows: Given the triangular covering map $p : \tilde{G} \to G$, consider a group $\Gamma \leqslant \mathrm{Aut}(\tilde{G})$ such that $p \circ \gamma = p$ for all $\gamma \in \Gamma$ (this means that all the fibers $p^{-1}(v)$ are Γ-invariant, so Γ acts on the fibers of p) then, if Γ acts transitively on the fibers of p, this p is a Galois triangular cover with group Γ. Indeed: the fibers of p are the Γ-orbits of the vertices of $\tilde{G}$, and by unique edge lifting $p^{-1}(v)$ and $p^{-1}(w)$ contain adjacent representatives if and only if v and w are adjacent in G, so G can be identified with $\tilde{G}/\Gamma$ and p can be thought of as p_Γ; the admissibility of Γ follows from Lemma 3.1.

Proposition 3.2. *Let $p : \tilde{G} \to G$ be Galois with group Γ. Then $p_k : k\tilde{G} \to kG$ is also Galois with group $\Gamma_k \cong \Gamma$.*

Proof. We know by Proposition 2.2 that p_k is a triangular covering map. Since Γ is admissible, it is coaffine and we already know that $\Gamma_k = \{\gamma_k \mid \gamma \in \Gamma\} \leqslant \mathrm{Aut}(kG)$ is also coaffine and $\Gamma_k \cong \Gamma$. If $\tilde{Q} \in V(k\tilde{G})$ and $\gamma_k \in \Gamma_k$, then $p_k \gamma_k(\tilde{Q}) = p\gamma(\tilde{Q}) = p(\tilde{Q}) = p_k(\tilde{Q})$, so Γ_k acts on the fibers of p_k. We show that these actions are transitive: Let $Q \in V(kG)$ and let $\tilde{Q}$ and $\widetilde{Q'}$ be two liftings of Q. Take a $v \in Q$ and let $\tilde{v}$ and $\widetilde{v'}$ be the liftings of v to $\tilde{Q}$ and $\widetilde{Q'}$, respectively. By hypothesis, there exists a $\gamma \in \Gamma$ such that $\gamma(\tilde{v}) = \widetilde{v'}$. Then $\gamma_k(\tilde{Q})$ is a lifting of Q which contains $\widetilde{v'} \in \widetilde{Q'}$, so $\gamma_k(\tilde{Q}) = \widetilde{Q'}$ and p_k is Galois with group Γ_k. $\square$

Proposition 3.2 can be rephrased as follows: If $\Gamma \leqslant \mathrm{Aut}(G)$ is admissible, then there exists an isomorphism $k(G/\Gamma) \cong (kG)/\Gamma_k$, and if $p_\Gamma : G \to G/\Gamma$ is the natural projection, then the triangular covering map $(p_\Gamma)_k : kG \to k(G/\Gamma)$ is Galois and gets identified by that isomorphism with the natural projection $p_{\Gamma_k} : kG \to (kG)/\Gamma_k$.

4. The orientable (abelian) case

Let G be a finite locally C_6 graph such that the triangular complex $\mathbb{K} = \mathbb{K}(G)$ is a torus $\mathbb{K} = \mathbb{S}^1 \times \mathbb{S}^1$. Let $\pi : \widetilde{\mathbb{K}} \to \mathbb{K}$ be the universal covering map, which has a covering space $\widetilde{\mathbb{K}}$ homeomorphic to the plane $\mathbb{R}^2$ and is regular with group $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$. The action of $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ in the plane $\mathbb{R}^2$ is by translations: if $\gamma = (a, b) \in \Gamma$ and

$x=(x_1,x_2)\in\mathbb{R}^2$, then $\gamma x=(a+x_1,b+x_2)$. As before, the graph G is embedded in $\mathbb{K}$ and we can lift it to a graph $\tilde{G}$ embedded in $\tilde{\mathbb{K}}=\mathbb{R}^2$. Since the fibers of π are infinite, $\tilde{G}$ is an infinite graph. We refer to $\tilde{G}$ as *the universal triangular cover* of G, and to the induced triangular covering map $p:\tilde{G}\to G$ as *the universal triangular covering map* for G. In the non-orientable (even in the infinite) case, the universal triangular covering map still has $\tilde{\mathbb{K}}=\mathbb{R}^2$, but we do not really need this for our result. The advantage of restricting to the orientable case will be that the Galois group is a group of translations. Due to the definition of $\tilde{G}$, the translations in Γ restrict to automorphisms of the graph $\tilde{G}$ and these restrictions act transitively on the fibers of $p:\tilde{G}\to G$, so p is a Galois triangular covering map with group Γ.

We describe now the graph $\tilde{G}$, which is the same for all locally C_6 graphs G. For this part, it is not essential that we are in the orientable (or finite) case. Let us denote by $\mathscr{T}$ the underlying graph of the tessellation of the Euclidean plane by equilateral triangles. In the complex plane, each vertex v of $\mathscr{T}$ has a unique expression as $v=a+b\omega$ where $a,b\in\mathbb{Z}$ and $\omega=\exp(\pi i/3)$ is the first primitive sixth root of 1; the edges of $\mathscr{T}$ are described as follows: if $v,w\in V(\mathscr{T})$, then $[v,w]\in E(\tilde{G})$ if and only if $||w-v||=1$. We call this the *geometrical description* of $\mathscr{T}$.

Proposition 4.1. *If G is a locally C_6 graph and $p:\tilde{G}\to G$ is the universal triangular covering map, then $\tilde{G}\cong\mathscr{T}$.*

Proof. We know already that $\tilde{G}$ is locally C_6. Let us fix an orientation of $\tilde{\mathbb{K}}=\mathbb{R}^2$: in particular, this gives us a cyclical ordering of the edges around each vertex of $\tilde{G}$. A *colorientation* at a vertex $\tilde{v}\in V(\tilde{G})$ consists in coloring the six edges containing $\tilde{v}$ with the colors a,b,c and orienting these edges in such a way that, in the fixed cyclical ordering, the edges around v appear in the cyclical order $(a,b,c,a^{-1},b^{-1},c^{-1})$, where by x^{-1} we denote an edge of color x directed towards $\tilde{v}$, whereas by x we mean the same color x but the opposite orientation. If $[\tilde{v},\tilde{w}]\in E(\tilde{G})$, a colorientation at $\tilde{v}$ and another at $\tilde{w}$ are *compatible* if they give the same color and the same orientation to the edge $[\tilde{v},\tilde{w}]$. A colorientation at a given vertex $\tilde{v}$ induces a unique compatible colorientation at any neighbor $\tilde{w}$ of $\tilde{v}$, and this colorientation in turn induces the original one at $\tilde{v}$. If $[\tilde{u},\tilde{v},\tilde{w}]$ is a triangle in $\tilde{G}$ and we start with a colorientation $\mathscr{C}(\tilde{u})$ at $\tilde{u}$, denote by $\mathscr{C}(\tilde{v})$ and $\mathscr{C}(\tilde{w})$ the colorientations induced by $\mathscr{C}(\tilde{u})$ at $\tilde{v}$ and $\tilde{w}$: then $\mathscr{C}(\tilde{w})$ is also the colorientation at $\tilde{w}$ induced by $\mathscr{C}(\tilde{v})$. Let us fix a base vertex $\widetilde{v_0}$ and a colorientation at $\widetilde{v_0}$; if $\tilde{\gamma}=(\widetilde{v_0},\widetilde{v_1},\dots,\widetilde{v_l})$ is any path, the colorientation at $\widetilde{v_0}$ induces one at $\widetilde{v_1}$, which induces one at $\widetilde{v_2}$, etc., and so the path $\tilde{\gamma}$ induces after l steps a colorientation in $\widetilde{v_l}$. If $\widetilde{\gamma'}$ is any other path from $\widetilde{v_0}$ to $\widetilde{v_l}$, the simple connectedness of $\tilde{\mathbb{K}}=\mathbb{R}^2$ ensures that γ and γ' are homotopic when considered as continuous curves, or that they are *triangularly homotopic* when considered just as paths. This last means that $\widetilde{\gamma'}$ can be obtained from $\tilde{\gamma}$ after a finite number of elementary transitions of the following kinds or their inverses: (i) if $\widetilde{v_i}$ and $\widetilde{v_{i+1}}$ are equal, omit $\widetilde{v_{i+1}}$, (ii) if $\widetilde{v_i}$ and $\widetilde{v_{i+2}}$ are equal, omit $\widetilde{v_{i+1}}$ and $\widetilde{v_{i+2}}$, and (iii) if $\widetilde{v_i}$, $\widetilde{v_{i+1}}$ and $\widetilde{v_{i+2}}$ form a triangle, omit $\widetilde{v_{i+1}}$. By our previous remarks, any path from $\widetilde{v_0}$ to $\widetilde{v_l}$ will induce the same colorientation at $\widetilde{v_l}$.

Since $\tilde{G}$ is connected, the fixed colorientation at $\widetilde{v_0}$ induces colorientations at all the vertices of $\tilde{G}$ in a compatible way: we say that the whole $\tilde{G}$ has been (compatibly) coloriented. Of course, this also proves that $\mathscr{T}$ can be coloriented, but this is obvious from the geometrical description.

Let us fix colorientations in both $\tilde{G}$ and $\mathscr{T}$. We define a coloriented graph $\vec{H}$ on the vertex set $V(\vec{H}) = V(\tilde{G}) \times V(\mathscr{T})$ by putting an arc $(u,u') \to (v,v')$ of color $x \in \{a,b,c\}$ if and only if there exist arcs $u \to v$ in $\tilde{G}$ and $u' \to v'$ in $\mathscr{T}$, both of the same color x. The underlying graph H of $\vec{H}$ is locally C_6 and both projections $H \to \tilde{G}$ and $H \to \mathscr{T}$ are triangular covering maps, from which it follows that their restrictions to any connected component C of H are also triangular covering maps. The continuous functions $\mathbb{K}(C) \to \mathbb{K}(\tilde{G})$ and $\mathbb{K}(C) \to \mathbb{K}(\mathscr{T})$ associated with these restrictions, being covering maps with simply connected codomains, are homeomorphisms, so we see that the restrictions themselves must be isomorphisms of graphs. Thus we conclude that $\tilde{G} \cong C \cong \mathscr{T}$. $\square$

By the preceding proposition, it will be useful to know the iterated clique graphs of $\mathscr{T}$, since any finite locally C_6 graph is of the form $\mathscr{T}/\Gamma$ for some admissible group $\Gamma \leqslant \mathrm{Aut}(\mathscr{T})$. In order to describe the iterated clique graphs of $\mathscr{T}$, we will use the following *non-geometrical* description: take $V(\mathscr{T}) = \mathbb{Z} \oplus \mathbb{Z}$; then, if we put $T = \{(0,0),(1,0),(0,1)\} \subset V(\mathscr{T})$ and $T - T = \{t - t' \mid t,t' \in T\}$, the adjacencies in $\mathscr{T}$ can be described as follows: if x and y are any two different vertices, then $[x,y] \in E(\mathscr{T})$ if and only if $y - x \in T - T$. The non-geometrical description corresponds to replacing the vertex $v = a + b\omega$ for the pair (a,b), so the translations still act by the addition of a constant pair. For instance, if $r,s \in \mathbb{Z}$ are both greater than 3 and Γ is the admissible subgroup of $\mathrm{Aut}(\mathscr{T})$ generated by the translations $x \mapsto x + (r,0)$ and $x \mapsto x + (0,s)$, then $\mathscr{T}/\Gamma$ is the graph $G(r,s)$, whose iterated clique graphs were completely described in [9]. The description of the iterated clique graphs of $\mathscr{T}$ is identical to that of the graphs $G(r,s)$ and the proofs can be read off directly from [9]. Alternatively, the results in [9] can be taken for granted and then the following description of the iterated clique graphs of $\mathscr{T}$ can be obtained by lifting.

Let $P = \{(1,1)\}$ and define the subsets E_j $(j \in \mathbb{Z})$ of $V(\mathscr{T})$ as follows: $E_0 = T - T$, $E_1 = T + T$, $E_2 = P + T$, $E_3 = P + P$, $E_j = \emptyset$ for $j > 3$, and $E_j = -E_{-j}$ for $j < 0$.

For any non-negative integer n, we will define a graph $\mathscr{T}^n$. It will turn out that $k^n\mathscr{T} \cong \mathscr{T}^n$. The vertex set of $\mathscr{T}^n$ is $V(\mathscr{T}^n) = \{n\} \times \{0,1,\ldots,n\} \times V(\mathscr{T})$. The more compact notation $x_i^n = (n,i,x)$ will be used. By definition, two different vertices x_i^n and y_j^n are adjacent in $\mathscr{T}^n$ if and only if $y - x \in E_{j-i}$.

If $i \in \{0,1,\ldots,n\}$, denote by $\mathscr{T}_i^n$ the subgraph of $\mathscr{T}^n$ induced by $\{x_i^n \mid x \in V(\mathscr{T})\}$. This graph $\mathscr{T}_i^n$ is isomorphic to $\mathscr{T}$, and $V(\mathscr{T}^n)$ is the disjoint union of the vertex sets of these $n+1$ copies of $\mathscr{T}$. In particular, for $n = 0$, $\mathscr{T}^0 = \mathscr{T}_0^0 \cong \mathscr{T}$.

In order to describe the cliques of $\mathscr{T}^n$, we use the following conventions: If $j \in \mathbb{Z}$ but $j \notin \{0,\ldots,n\}$, we put $\mathscr{T}_j^n = \emptyset$. If $X \subseteq V(\mathscr{T})$, let $X_j^n = \{x_j^n \in V(\mathscr{T}_j^n) \mid x \in X\}$; in particular, $X_j^n = \emptyset$ if $j \notin \{0,\ldots,n\}$.

If $i \in \{0,\dots,n+1\}$ and $x \in V(\mathscr{T})$, put

$$Q^n_{i,x} = (x-P)^n_{i-2} \cup (x-T)^n_{i-1} \cup (x+T)^n_i \cup (x+P)^n_{i+1}.$$

These $Q^n_{i,x}$ are precisely the cliques of $\mathscr{T}^n$, and $\varphi_n(Q^n_{i,x}) = x^{n+1}_i$ defines an isomorphism of graphs $\varphi_n : k\mathscr{T}^n \to \mathscr{T}^{n+1}$.

We have an isomorphism $\psi_0 : \mathscr{T} \to \mathscr{T}^0$. Then $\psi_1 = \varphi_0 \circ (\psi_0)_k : k\mathscr{T} \to \mathscr{T}^1$ is an isomorphism and inductively we have, for each non-negative n, an isomorphism $\psi_{n+1} = \varphi_n \circ (\psi_n)_k : k^{n+1}\mathscr{T} \to \mathscr{T}^{n+1}$. Hence, the nth iterated clique graph of $\mathscr{T}$ is isomorphic to our graph $\mathscr{T}^n$.

Proof of Theorem 1.1. We know already that we can restrict to the case in which the associated surface $\mathbb{K} = \mathbb{K}(G)$ is a torus, and that G is of the form $G = \mathscr{T}/\Gamma$ for some admissible group $\Gamma \leqslant \mathrm{Aut}(\mathscr{T})$. Let us use the non-geometrical coordinatization $V = \mathbb{Z} \oplus \mathbb{Z}$ of the vertices of $\mathscr{T}$. Any translation $\gamma \in \Gamma$ is of the form $x \mapsto x + g_\gamma$ for some $g_\gamma \in \mathbb{Z} \oplus \mathbb{Z}$, and the set of all these g_γ forms a subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ which is isomorphic to Γ. Recall that $kG = k(\mathscr{T}/\Gamma) \cong (k\mathscr{T})/\Gamma_k$, where $\Gamma_k = \{\gamma_k \mid \gamma \in \Gamma\} \leqslant \mathrm{Aut}(k\mathscr{T})$, $\gamma_k(Q) = \gamma(Q)$ for all $Q \in V(k\mathscr{T})$, and $\Gamma_k \cong \Gamma$.

Let $\Gamma_{k^{n+1}}$ be the admissible group $\Gamma_{k^{n+1}} = \{\gamma_{k^{n+1}} \mid \gamma_{k^n} \in \Gamma_{k^n}\} \leqslant \mathrm{Aut}(k^{n+1}\mathscr{T})$, where $\gamma_{k^{n+1}}(Q) = \gamma_{k^n}(Q)$ for all $Q \in V(k^n\mathscr{T})$. We have that $k^nG \cong (k\mathscr{T}^n)/\Gamma_{k^n}$ for all n and, since $\Gamma_{k^n} \cong \Gamma$ for all n, what we have is an action of Γ in each $k^n\mathscr{T}$ in such a way that k^nG is the quotient graph of $k^n\mathscr{T}$ under this action of Γ.

On the other hand, since Γ is a group of translations, there is a natural action of Γ in each $\mathscr{T}^n$: if $x^n_i \in V(\mathscr{T}^n)$ and $\gamma \in \Gamma$, we put $\gamma(x^n_i) = (\gamma x)^n_i$. Notice that this action restricts to each of the $n+1$ subgraphs $\mathscr{T}^n_i$, and that $\mathscr{T}^n_i$ is isomorphic to $\mathscr{T}$ in a Γ-equivariant way.

In particular, the isomorphism $\psi_0 : \mathscr{T} \to \mathscr{T}^0$ which sends x to x^0_0 is Γ-equivariant, and it follows by induction on n that all the $\psi_{n+1} = \varphi_n \circ (\psi_n)_k : k^{n+1}\mathscr{T} \to \mathscr{T}^{n+1}$ are Γ-equivariant. Indeed, $\psi_n : k^n\mathscr{T} \to \mathscr{T}^n$ is equivariant by the inductive hypothesis, but then $(\psi_n)_k : k^{n+1}\mathscr{T} \to k\mathscr{T}^n$ is also equivariant because for any $Q \in V(k^{n+1}\mathscr{T})$ and $\gamma \in \Gamma$ we have $(\psi_n)_k(\gamma Q) = \psi_n(\gamma Q) = \gamma\psi_n(Q) = \gamma((\psi_n)_k(Q))$. Then it is enough to establish the equivariance of $\varphi_n : k\mathscr{T}^n \to \mathscr{T}^{n+1}$, but for any translation γ we have that $\varphi_n(\gamma Q^n_{i,x}) = \varphi_n(Q^n_{i,\gamma x}) = (\gamma x)^{n+1}_i = \gamma(x^{n+1}_i) = \gamma\varphi_n(Q^n_{i,x})$.

Now we have that k^nG is isomorphic to the quotient of $\mathscr{T}^n$ under the action of Γ, and recalling that $V(\mathscr{T}^n)$ is the disjoint union of the vertex sets of the $n+1$ subgraphs $\mathscr{T}^n_i$, that these are Γ-invariant, and that $\mathscr{T}^n_i/\Gamma \cong \mathscr{T}/\Gamma \cong G$, we deduce that the vertex set of $k^nG = \mathscr{T}^n/\Gamma$ is also the disjoint union of the vertex sets of the $n+1$ subgraphs $G_i = \mathscr{T}^n_i/\Gamma$, each isomorphic to G. Thus we obtain that G is k-divergent and that, furthermore, the order $\mathrm{o}(k^nG) = (n+1) \cdot \mathrm{o}(G)$ is a linear polynomial in n. Since the furthest copy G_j of G in k^nG that can be reached by an edge with an end in G_i is G_{i+3}, we see that $\lfloor (n+1)/3 \rfloor \leqslant d(k^nG)$, and by Corollary 2.3(3) and the description of the cliques of $\mathscr{T}$ we get that $\omega(k^nG) = 8$ for all $n \geqslant 3$. $\square$

5. Final remarks

The title of the preceding section contains a reference to our first proof, which was based on fundamental groups. The *triangular fundamental group* of G is just the edge-path group of the triangular complex $\mathbb{K}(G)$ (or, equivalently, of the total complex $G^{\uparrow}$); thus, if we fix a base vertex $v_0 \in V(G)$, $\pi_1(G, v_0)$ is the set of all triangular homotopy classes (see the proof of Proposition 4.1) of closed paths at v_0 with the obvious composition, and if we choose another base vertex v_1 by connectedness we have $\pi_1(G, v_1) \cong \pi_1(G, v_0)$ and we can denote this group simply by $\pi(G)$. If we choose a base clique Q_0 containing the base vertex v_0, then a trivial calculation shows that $\pi_1(G, v_0) \cong \pi_1(kG, Q_0)$ and hence the triangular fundamental group is k-invariant. We mention three consequences:

1. Since the triangular fundamental group of the trivial graph is the trivial group, any k-null graph G is *triangularly simply connected*, i.e., $\pi_1(G) = 1$ (cf. [14, Corollary 4.1]).
2. Since the universal triangular covering map of a given graph G can be characterized as the only triangular covering map $p : \tilde{G} \to G$ with triangularly simply connected domain, it follows that in this case $p_k : k\tilde{G} \to kG$ is also a universal triangular covering map.
3. The classes of graphs $\mathscr{C}$ such that $k\mathscr{C} \subseteq \mathscr{C}$ have been called k-semibasins by E. Prisner (see [15,16]). Since the triangular fundamental group is k-invariant, for each isomorphism class of finitely presented groups $[\Gamma]$ we obtain a non-empty semibasin $\mathscr{C}_\Gamma$ consisting of all graphs G with $\pi_1(G) \cong \Gamma$ (see [18, 11.65] but note that the triangles $\{w, u^x, v^x \mid x \in X\}$ in the proof of [18, 11.64] need to be replaced by squares $\{w, t^x, u^x, v^x \mid x \in X\}$), and the class of all graphs is the disjoint union of these k-semibasins.

We have mentioned in the introduction that the k-divergence of the only locally C_5 graph is an open problem. Since (among other difficulties) the triangular complex $\mathbb{K}$ for the icosahedron is the sphere $\mathbb{S}^2$ and this is simply connected, the covering techniques used in this paper do not help to attack that problem. On the other hand, for each $t > 6$ finite locally C_t graphs do abundantly exist [2] and for them our method seems to be helpful. The Euler characteristic of the surface $\mathbb{K}$ is negative and no longer a function of t alone, but of the order too. The universal triangular cover, to which the study ultimately reduces, is the 1-skeleton of the tessellation of the hyperbolic plane by equilateral triangles with interior angle $2\pi/t$. It seems that the combinatorial part, analogous to the work in [9], is what is lacking.

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